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Parallel Planar Subgraph Isomorphism and Vertex Connectivity
**Subgraph Isomorphism:**
Find subgraphs in the target that *match the pattern*.

**Target** $G$
- $n$ vertices

**Pattern** $H$
- $k$ vertices

Diagram:
- Vertices: $a, b, c, d, e, f, g, h, i, j$
- Edges: $ab, bc, cd, de, ef, fg, gh, hi, ij$

Question mark indicating where the pattern needs to be matched.
Subgraph Isomorphism:
Find subgraphs in the target that **match the pattern**

Target $G$

$\text{n vertices}$

Pattern $H$

$\text{k vertices}$

2 occurrences
Subgraph Isomorphism:
Find subgraphs in the target that match the pattern

Target $G$
$n$ vertices

Planar target: NP-Hard

Pattern $H$
$k$ vertices

2 occurrences
Subgraph Isomorphism:
Find subgraphs in the target that match the pattern

Planar target: NP-Hard
Focus on small patterns

Target $G$
$n$ vertices

Pattern $H$
$k$ vertices

2 occurrences
Subgraph Isomorphism

Results for Planar Graphs

Color Coding
Alon et al. 1995

\( \Omega \left( n^{\sqrt{k}} \right) \)

\( O \left( \log^2 n \right) \)

\( \text{Work} \)
\( \text{Depth} \)

\(*\text{Result correct with high probability}*\)

\( n \) vertices

\( k \) vertices

Pattern \( H \)

Target \( G \)
### Results for Planar Graphs

<table>
<thead>
<tr>
<th>Color Coding*</th>
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<tbody>
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*Alon et al. 1995

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<td>$O\left(k^{3k+1}n\right)$</td>
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<td>$\Omega\left(n\right)$</td>
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Target $G$ with $n$ vertices

Pattern $H$ with $k$ vertices

Subgraph Isomorphism

*Result correct with high probability
Subgraph Isomorphism

Results for Planar Graphs

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<tr>
<td><strong>Our Result</strong></td>
<td>$O(k^{3k+1}n \log n)$</td>
</tr>
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</table>

* Result correct with high probability
Dynamic Programming
Dynamic Programming

“Shared Vertices” divide the graph
Dynamic Programming

Solve all subproblems for both parts

Pattern $H$

Partial Solution 1

Pattern $H$

Partial Solution 2

G

Dynamic Programming

Solve all subproblems for both parts

Pattern $H$

Partial Solution 1

Pattern $H$

Partial Solution 2

G
Dynamic Programming

Combine compatible partial solutions

Pattern $H$

Partial Solution 1

Partial Solution 2

$G$
Dynamic Programming

Exponential in “shared” part

General $\Omega(n)$
Dynamic Programming

Exponential in “shared” part

General \( \Omega(n) \)

Planar \( \Theta(\sqrt{n}) \)
Dynamic Programming

- Exponential in “shared” part
- General: $\Omega(n)$
- Planar: $\Theta(\sqrt{n})$
- Planar, diameter $d$: $O(d)$
Dynamic Programming

- Exponential in “shared” part
- General: $\Omega(n)$
- Planar: $\Theta(\sqrt{n})$
- Planar, diameter $d$: $O(d)$

Check diameter $k-1$ subgraphs
Naïve Covering

$G'$
Naïve Covering

$G'$
Naïve Covering

$G'$
Naïve Covering

$G'$
Naïve Covering

$G'$

$\Theta(n^2)$ work
Work-Efficient Covering with BFS

BFS Tree
Work-Efficient Covering with BFS

BFS Tree

$G'$

$G_0$
Work-Efficient Covering with BFS

BFS Tree
Work-Efficient Covering with BFS

$G'$

$G_0$

$G_1$

$G_2$

BFS Tree
Work-Efficient Covering with BFS

BFS Tree

$G'\, G_0\, G_1\, G_2$

$O(kn)$ work
Work-Efficient Covering with BFS

Problem: $\Omega(n)$ depth

$G'$

$G_0$

$G_1$

$G_2$

$O(kn)$ work

BFS Tree
Low-Diameter Decomposition
Miller et al. 2015

Target $G$

$n$ vertices
Low-Diameter Decomposition
Miller et al. 2015

Cluster Diameter $O(k \log n)$

$n$ vertices

Target $G$
Low-Diameter Decomposition
Miller et al. 2015

Cluster Diameter
\(O(k \log n)\)

\(n\) vertices

Target \(G\)

Probability a particular edge crosses \(\leq \frac{1}{2k}\)
**Low-Diameter Decomposition**
Miller et al. 2015

- **Cluster Diameter** \( O(k \log n) \)
- **n vertices**
- **Target** \( G \)
- **Pattern** \( H \)
- **k vertices**

**Probability a particular edge crosses** \( \leq \frac{1}{2k} \)

**Probability an occurrence crosses** \( \leq \frac{1}{2} \)
Low-Diameter Decomposition
Miller et al. 2015

Cluster Diameter \( O(k \log n) \)

\( n \) vertices

Target \( G \)

Probability a particular edge crosses \( \leq \frac{1}{2k} \)

Probability an occurrence crosses \( \leq \frac{1}{2} \)

Pattern \( H \)

\( k \) vertices

Pattern \( H \)
Planar Subgraph Isomorphism

Low Diameter Decomposition

- \( O(n) \) work
- \( O(k \log n) \) depth

Covering with BFS

- \( O(kn) \) work
- \( O(k \log n) \) depth

Dynamic Programming

- \( O(k^{3k+1}n) \) work
- \( O(k \log n) \) depth

\( O(\log n) \) repetitions
Subgraph Isomorphism

Target $G$

Pattern $H$

Minimum Vertex Cut

$G$
Minimum Vertex Cut
Smallest *number of vertices* whose *removal disconnects the graph*
Minimum Vertex Cut

$G$

$G'$

Face Vertices

Original Vertices
Minimum Vertex Cut → Separating Cycle

$G$ → $G'$

- Face Vertices
- Original Vertices
Minimum Vertex Cut

Separating Cycle

$G$

$G'$

Constant Length

$O(n \log n)$ work

$O(\log n)$ depth

Face Vertices

Original Vertices

$\mathcal{O}(n \log n)$ work
**Subgraph Isomorphism**

- Target $G$
- $n$ vertices
- Pattern $H$
- $k$ vertices
- $O(k^{3k+1}n \log n)$ work
- $O(k \log n)$ depth

**Minimum Vertex Cut**

- $G'$
- $O(n \log n)$ work
- $O(\log n)$ depth

**Conclusion**
Conclusion

**Subgraph Isomorphism**

- Target $G$ with $n$ vertices
- Pattern $H$ with $k$ vertices
- **Work**: $O(k^{3k+1}n \log n)$
- **Depth**: $O(k \log n)$

**Minimum Vertex Cut**

- **Work**: $O(n \log n)$
- **Depth**: $O(\log n)$

Questions:
- Singly exponential in $k$?
- Linear in $n$?
- Polylog in $k$?
**Conclusion**

### Subgraph Isomorphism

- **Target** $G$
- **Pattern** $H$
- **$k$ vertices**
- **$n$ vertices**

- **Work:** $O(k^{3k+1}n \log n)$
- **Depth:** $O(k \log n)$

- **Singly exponential in $k$?**
- **Polylog in $k$?**
- **Other Implications?**

### Minimum Vertex Cut

- **$G$**
- **$G'$**
- **Minimum Vertex Cut**
- **Separating Cycle**
- **Face Vertices**
- **Original Vertices**

- **Work:** $O(n \log n)$
- **Depth:** $O(\log n)$

- **Linear in $n$?**