Abstract—In this paper we present PolarFly, a diameter-2 network topology based on the Erdős-Rényi family of polarity graphs from finite geometry. This is a highly scalable low-diameter topology that asymptotically reaches the Moore bound on the number of nodes for a given network degree and diameter. PolarFly achieves high Moore bound efficiency even for the moderate radixes commonly seen in current and near-future routers, reaching more than 96% of the theoretical peak. It also offers more feasible router degrees than the state-of-the-art solutions, greatly adding to the selection of scalable diameter-2 networks. PolarFly enjoys many other topological properties highly relevant in practice, such as a modular design and expandability that allow incremental growth in network size without rewiring the whole network. Our evaluation shows that PolarFly outperforms competitive networks in terms of scalability, cost and performance for various traffic patterns.

I. INTRODUCTION

Traditional demand for scalable networks comes from government labs and research institutions to perform large scientific simulations. For example, Fukagami [1], the largest supercomputer in the world at the time of this writing, connects simulation modules to perform large scientific simulations. For example, Fukagami [1], the largest supercomputer in the world at the time of this writing, connects simulation modules to perform large scientific simulations. For example, Fukagami [1], the largest supercomputer in the world at the time of this writing, connects simulation modules to perform large scientific simulations. For example, Fukagami [1], the largest supercomputer in the world at the time of this writing, connects simulation modules to perform large scientific simulations. For example, Fukagami [1], the largest supercomputer in the world at the time of this writing, connects simulation modules to perform large scientific simulations. For example, Fukagami [1], the largest supercomputer in the world at the time of this writing, connects simulation modules to perform large scientific simulations. For example, Fukagami [1], the largest supercomputer in the world at the time of this writing, connects simulation modules to perform large scientific simulations. For example, Fukagami [1], the largest supercomputer in the world at the time of this writing, connects simulation modules to perform large scientific simulations. For example, Fukagami [1], the largest supercomputer in the world at the time of this writing, connects simulation modules to perform large scientific simulations. For example, Fukagami [1], the largest supercomputer in the world at the time of this writing, connects simulation modules to perform large scientific simulations. For example, Fukagami [1], the largest supercomputer in the world at the time of this writing, connects simulation modules to perform large scientific simulations.

II. RELATED WORK

A. State of Art Diameter 2 Topologies: Slim Fly

Given the availability of high-radix routers, it is desirable to maximize the number of nodes that can be supported on a network of a given diameter. Slim Fly [26] was the first topology analyzed in the networking community that explicitly optimized its structure towards the Moore bound [27], an upper
bound on the number of vertices in a graph with given diameter and for a given maximal degree. By fixing its diameter to 2, Slim Fly reduces construction cost and power consumption, while ensuring low latency and high bandwidth.

However, Slim Fly has several issues with respect to practical layout and deployment. The number of feasible configurations/topological constructions is limited, and it is not competitive with commercially available products. And there are no results in the literature to address network re-configurability and expansion. This is a matter of great importance in real-world scenarios, where data centers need to increase the size of a compute center gradually over time without being forced to rewire and re-layout the whole interconnect.

Fig. 1: Design space of feasible degrees (network radixes) for PolarFly and Slim Fly. Asymptotically, there are 50% more PolarFly feasible degrees than what Slim Fly offers.

II. BACKGROUND

A. Network Model

We model an interconnection network as an undirected graph $G = (V,E)$; $V$ is the set of nodes and $E$ is the set of links ($|V| = N$). We consider only direct networks with co-packaged modules, so there is no notion of endpoints attached to routers: each node in the network serves both as a router/switch and as a compute endpoint. There are $N$ such nodes in total, and $k$ channels from each node to other node (radix). The diameter $D$ denotes the maximum length of shortest paths between any pair of nodes.

B. The Degree-Diameter Problem and Network Design

The degree of a network is determined by current technology, and the diameter is chosen according to the system requirements. Based on this, one would like to maximize the number of nodes in such a network. This is the degree-diameter problem: find the maximum number $n(D, k)$ (or $N$) of vertices in a graph given maximal degree $k$ and diameter $D$.

The degree-diameter problem is a major open problem in graph theory. For a comprehensive survey, see [28]. Bounds exist for this problem, but few optimal graphs have been identified. Loz, Pérez-Rosés and Pineda-Villavicencio give two tables [27] with the largest known graphs and bounds as of 2010 for a given degree and diameter. Only a few of the graphs are known to be optimal.

1) The Moore Bound

The Moore bound [29] is the most general upper bound on the number of vertices $n$ for a graph with maximum degree $k$ and diameter $D$, and is given by

$$N \leq 1 + k \cdot \sum_{i=0}^{D-1} (k - 1)^i. \quad (1)$$

Few graphs of any diameter and degree actually meet the Moore bound; in fact, few even come close. Hoffman and Singleton [29], Bannai and Ito [30], and Damerell [31] have identified all of the graphs that meet the bound.

The Erdős-Rényi polarity graphs were introduced by Erdős and Rényi in [32] and by Brown in [33]. They have diameter 2 and asymptotically approach the Moore bound, which is $N \leq 1 + k^2$ for graphs of diameter 2. They also have properties useful for network design, which we exploit for PolarFly.
III. Feasibility Analysis of Candidate Topologies

There are many available topologies. However, not all are suitable for use in a data center, especially in a co-packaged setting. In this section, we investigate representative networks and show that PolarFly meets the data center needs best of all.

We consider Slim Fly [26] (a variant with \( D = 2 \)), Dragonfly [34] (the “balanced” variant with \( D = 3 \)), and HyperX (Hamming graph) [35] that generalizes Flattened Butterflies [36] with \( D = 2 \). We also use established three-stage Fat Trees [37]. Finally, for completeness, we also consider two Fat Tree variants, Orthogonal Fat Trees [38] and Multi-Layer Full Meshes [38]. In the following, we identify the criteria for a topology to be a suitable candidate for a data center.

Directness. Direct networks can be constructed using only one type of co-packaged chip that integrates the compute, routing hardware, and communication ports in the same package. In contrast, indirect networks such as fat trees, require design, fabrication, and deployment of additional chiplet(s) for the switches, which significantly increases their overall cost.

Flexibility. A flexible network provides many feasible configurations that could be constructed using available equipment while delivering high performance. This means that one must be able to build networks using switches with feasible radix.

Low Diameter. Upcoming distributed shared-memory systems such as PIUMA [39], and future disaggregated memory systems [40], heavily rely on low-latency remote accesses for performance scalability. This can only be delivered by scalable networks with small diameter, ideally two, or networks with average path length of two. In case of direct networks, low-diameter topologies also support higher ingestion bandwidth.

Modularity. In a modular network, the nodes can be decomposed into smaller units that could be, e.g., racks, blades, or chassis. This feature facilitates manufacturing, deployment and cabling. Most of the considered networks satisfy this requirement. For example, plain Fat Trees consist of pods, while Dragonflies have a group-based structure.

Expandability. A network is expandable if its size can be incrementally increased by adding a basic unit, such as a rack, by using empty ports in an under-provisioned network. This need is usually related to budgetary issues – the budget-limited purchased system is smaller than the optimal system, and may only be extended to a larger size later. Incremental growth may be preferred over complete rewiring into a new topology, as the latter is much more disruptive, expensive and time-consuming. While some of the considered networks, like Dragonfly, do enable incremental growth, it is not known how to increase the size of the most competitive target Slim Fly.

We now analyze the considered networks and show whether they satisfy the above criteria. A summary of the analysis is illustrated in Table I. All networks are at least partially modular and flexible. Most networks have diameter two. Only PolarFly satisfies all the criteria almost fully.

IV. POLARFLY TOPOLOGY

In this section, we discuss in detail the graph underlying the PolarFly layout. This mathematical description is used in the construction and for the exploitation of the graph properties.

The topology of PolarFly is an Erdős-Rényi (ER) polarity graph, also known as a Brown graph, constructed using the relationship of points and lines in finite geometry. These were discovered independently by Erdős and Rényi [32] and Brown [33]. There is a great deal of mathematical structure to these graphs, and they have been studied in depth, both in the original papers [32], [33] and other references, e.g. [41]–[43]. ER graphs have several useful features for network design:

- **Low diameter.** They have diameter 2, giving a short path between any two nodes.
- **Scalability.** At the same time, they asymptotically reach the Moore bound, surpassing the scalability of all other diameter-2 topologies. We compare to other diameter-2 topologies that are direct, as needed for co-packaging, and show the comparison in Figure 2.
- **Flexibility.** They cover a wide range of degrees, having degree \( k = q + 1 \) for every prime power \( q \). While not completely general, they meet or come very close to the radixes of many current and near-term high-radix routers. For example, for \( q = 31, 47, 61 \) and 127, \( ER_q \) may be applied to systems with routers of radix 32, 48, 64 and 128, with all router ports used at radix 32, 48 and 128.

![Moore Bound Comparison of Diameter-2 Topologies](image)

**Fig. 2:** Scalability of direct diameter-2 topologies in terms of the optimal Moore bound. Petersen (10 vertices) and Hoffman-Singleton (50 vertices) are the only known graphs to achieve the Moore Bound.

A. Some Background on Finite Fields and Their Arithmetic

The construction of ER polarity graphs is based upon the arithmetic operations of finite fields. A field is a set having addition and multiplication, where every element has an additive inverse, and every non-zero element has a multiplicative inverse. The construction of ER polarity graphs depends especially upon the existence of multiplicative inverses.

The set of integers modulo a prime \( p \) is an example of a finite field. Finite fields \( \mathbb{F}_q \) of order \( q \) exist for all prime

<table>
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<tr>
<th>Topology</th>
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<th>Expandable</th>
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<th>Diameter-2</th>
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**TABLE I:** Feasibility. "●": full support, "○": partial support, "✗": no support.
powers \( q \), and for no other integers. Finite fields are also called Galois extension fields. They are fundamental to many areas in mathematics and computer science, and are discussed in detail in [44]–[46] and in many other references.

It is important to note that addition and multiplication operations in \( \mathbb{F}_q \) are quite different from those in \( \mathbb{R} \):

- If \( q = p \) is a prime, then addition and multiplication are just modular arithmetic over \( p \).
- If \( q = p^m \) is a prime power, with \( m > 1 \), then addition and multiplication in \( \mathbb{F}_q \) are derived from modular arithmetic over an irreducible degree-\( m \) polynomial over \( \mathbb{F}_p \).

For further details on arithmetic in the finite fields \( \mathbb{F}_q \), with \( q \) not prime, see any of the references listed above.

We will use primes \( q = p \) for the examples in this paper for simplicity, so the addition and multiplication in the dot products is just modular arithmetic over \( p \). The same graph construction will hold for any prime power \( q = p^m \), using the associated arithmetic of \( \mathbb{F}_q \) for the dot products.

### B. Geometric Intuition

Erdős-Rényi polarity graphs express the orthogonality (or perpendicularity) between vectors, or equivalently, lines passing through the origin. The dot product is a convenient way of expressing this; two length-\( n \) vectors \( v \) and \( w \) are orthogonal when \( v \cdot w = \sum_{i=1}^{n} v_i w_i = 0 \).

Note that multiples of a vector retain the same orthogonality relationships as the original vector. So for our purposes, we may consider all multiples of a vector to be the same, and simply choose one as representative of all its multiples.

ER polarity graphs are defined by vertices that represent length-3 vectors over a field, and edges that exist between two vertices if the vectors they represent are orthogonal. This graph has diameter 2.

As an example, consider ordinary Euclidean 3-dimensional space. The existence of 2-hop paths between any two vectors in the corresponding graph depends upon this fact: any pair of (non-multiple) vectors has a vector to which both are orthogonal: their cross-product. The 2-hop path linking them passes through the orthogonal vector. This is shown in Figure 3.

![Fig. 3: Let \( \ell_0 \) and \( \ell_1 \) be arbitrary lines in Euclidean 3-space passing through the origin. The line \( \ell_0 \) is perpendicular to the plane \( P_0 \), and the line \( \ell_1 \) is perpendicular to the plane \( P_1 \). The two planes intersect in a line \( m \) which is perpendicular to both \( \ell_0 \) and \( \ell_1 \). A graph including \( \ell_0 \), \( \ell_1 \) and \( m \) as vertices has edges (\( \ell_0, m \)) and (\( m, \ell_1 \)), so there is a 2-hop path from \( \ell_0 \) to \( \ell_1 \) passing through \( m \). This construction may be generalized to \( \mathbb{F}_p^3 \), using the dot product to represent perpendicularity.](image)

Euclidean 3-space obviously has infinitely many lines and planes passing through the origin. However, a similar construction may be used to obtain a finite space of dimension 3 over the finite field \( \mathbb{F}_q \). The geometric relationships are similar to those discussed above in the Euclidean case. Orthogonality is again expressed by the dot product, using the addition and multiplication from \( \mathbb{F}_q \). So the construction over \( \mathbb{F}_q \) also gives rise to a graph of diameter 2, but this time the graph is finite.

Because \( \mathbb{F}_q \) is finite with modular arithmetic, some non-zero vectors in \( \mathbb{F}_q^3 \) have the interesting property that they are orthogonal to themselves, which never happens in Euclidean space. For example, consider \( \mathbb{F}_3^3 \) where the arithmetic operations are modular addition and multiplication \( \text{mod} \ 3 \). The vector \([1,1,1]\) is self-orthogonal, since \([1,1,1] \cdot [1,1,1] = 1 + 1 + 1 = 0 \) \( \text{mod} \ 3 \).

### C. Construction With Dot Products over \( \mathbb{F}_q \)

ER polarity graphs are easily constructed using the set of non-zero left-normalized vectors \([x, y, z] \in \mathbb{F}_p^3 \) as vertices. These are vectors in which the first non-zero entry is 1.

For example, in \( \mathbb{F}_3^3 \), \([1,0,2]\) and \([0,1,0]\) are vectors in the set under consideration, but \([0,2,1]\) would not be, since its first non-zero entry is 2. Instead, \([0,2,1]\) is multiplied by the multiplicative inverse of 2 \( \text{mod} \ 3 \), giving its left-normalized representation: \([2,0,1]\) \( \equiv [0,1,2] \) \( \text{mod} \ 3 \). The existence of multiplicative inverses in the field \( \mathbb{F}_q \) assu res us that each non-zero vector can be represented as a left-normalized vector.

\( ER_q \) is then constructed as follows:

- The vertices are the left-normalized vectors in \( \mathbb{F}_q^3 \).
- The edges are pairs \((v, w)\) of vertices that are orthogonal to each other, as per the dot product; in other words, using the addition and multiplication of \( \mathbb{F}_q \), the dot product of \( v \) and \( w \) is 0.

Self-orthogonal vertices are distinguished from the others and are called quadrics. All other vertices are called non-quadrics. Quadrics may be considered to have a self-loop, and play a special role in the construction of PolarFly.

We show an example of this construction in Figure 4. There is a great deal of structure in the \( ER_q \) graph, some of which can be seen there and in the graph layout in Figure 6. We exploit this in the construction of an efficient network.

![Fig. 4: The dot-product construction of \( ER_3 \). The left-normalized vectors of \( \mathbb{F}_3^3 \) are the vertices, arranged lexicographically, and clockwise starting at the top. The base field is \( \mathbb{F}_3 \). 3 is a prime rather than a prime power, so the operations are addition and multiplication \( \text{mod} \ 3 \). Edges exist between vertices \( v \) and \( w \) when the dot product \( v \cdot w \) is 0, using arithmetic from \( \mathbb{F}_3 \). For example, the vertex \([1,1,1]\) is adjacent to \([0,1,2]\), since the dot product \([1,1,1] \cdot [0,1,2] = 0 + 1 + 2 \equiv 0 \) \( \text{mod} \ 3 \). The self-adjacent quadrics \((V_q)\) for which the dot products \( w \cdot w \) are 0 are red. Vertices adjacent to quadrics \((V_1)\) are green, and vertices not adjacent to quadrics \((V_2)\) are blue.](image)
D. Minimal paths, intermediate points and routing

Since an ER graph has diameter 2, the minimal path between two vertices is either 1 or 2 hops. A path is of length 1 if and only if the vertices are orthogonal (so their dot product is 0).

There is only one minimal path of length 1 or 2 between vertices. For reasons of efficiency, table-based routing is the best method for finding paths. However, the unique intermediate vertex on a 2-hop path may also be found by solving a pair of linear equations representing the dot-product construction.

Intuitively, the two vertices are represented by two distinct lines, which are orthogonal to a single line (or single vertex in projective space), as seen in Figure 3. So the problem of finding the intermediate vertex in a minimal 2-hop path reduces to finding that unique orthogonal line.

The intermediate vertex \( v \) between vertices \( s \) and \( d \) is orthogonal to both \( s \) and \( d \), so \( s \cdot v = d \cdot v = 0 \). Thus \( v \) is found by solving this system of equations, via an augmented matrix:

\[
\begin{bmatrix}
  s_0 & s_1 & s_2 & 0 \\
  d_0 & d_1 & d_2 & 0
\end{bmatrix}
\]

Since \( s \) and \( d \) are not multiples of each other, and since all vectors in \( ER_q \) must be left-normalized, there will be a unique solution \( v \) to this system of equations, which may be found by Gaussian elimination or otherwise.

For example, in \( ER_3 \), the vectors \((0, 0, 1)\) and \((1, 2, 2)\) are not orthogonal, (since their dot product is not 0) so the minimal path has length 2. The left-normalized solution in \( \mathbb{F}_3 \) to the resulting augmented matrix

\[
\begin{bmatrix}
  0 & 0 & 1 & 0 \\
  1 & 2 & 2 & 0
\end{bmatrix}
\]

is \((1, 1, 0)\). Figure 4 confirms that \((1, 1, 0)\) is indeed intermediate on the unique length-2 path between \((0, 0, 1)\) and \((1, 2, 2)\).

Another, perhaps simpler way to obtain the vertex \( v \) orthogonal to both \( s \) and \( d \) is to compute the cross-product of \( s \) and \( d \), as discussed in [43]:

\[
s \times d = (s_2d_3 - d_3s_2, s_3d_1 - d_3s_1, s_1d_2 - s_2d_1)
\]

Since multiples of a vector represent the same vertex, the coordinates obtained from cross-product can be left-normalized to obtain \( v \). For example, in \( ER_3 \), the intermediate vertex between the vectors \((0, 0, 1)\) and \((1, 2, 2)\) is given by:

\[-2, 1, 0) = (1, 1, 0)\]

as vectors modulo 3.

Because vectors in \( ER_q \) are left-normalized, this method is quite efficient, in the worst case needing only two multiplies and three adds in \( \mathbb{F}_q \) to compute the cross-product, then at most another two multiplies for the left-normalization.

E. Formal Construction

1) A Bipartite Graph From Finite Geometry

The projective plane \( \text{PG}(2, q) \) is a geometric structure that arises from projecting lines and planes in three-dimensional space over \( \mathbb{F}_q \) to points and lines respectively. As discussed above, projective points can be thought of as the left-normalized vectors in \( \mathbb{F}_q^3 \). In more formal language, each point \([a] \) in \( \text{PG}(2, q) \) is an equivalence class of 3-tuples where \((a_1, a_2, a_3) \sim (b_1, b_2, b_3)\) if and only if these tuples are multiples of each other. Projective lines \((b_1 : b_2 : b_3)\) contain all points \([x] \) so that \(b_1x_1+b_2x_2+b_3x_3 = 0 \) in \( \mathbb{F}_q \). By counting such vectors, we see there are \(q^2 + q + 1\) points in \( \text{PG}(2, q) \). Dually, there are \(q^2 + q + 1\) lines in \( \text{PG}(2, q) \) as we will see.

We will construct \( ER_q \) graphs for any prime power \( q \) using properties of points and lines in \( \text{PG}(2, q) \). To begin, we first build a bipartite graph \( B(q) \). The vertex set of \( B(q) \) is \( U \cup V \) where \( U \) is the set of points in \( \text{PG}(2, q) \) and \( V \) is the set of lines in \( \text{PG}(2, q) \). There is an edge between \( v \in U \) and \( w \in V \) if and only if the point \( v \) lies on the line \( w \). In \( \text{PG}(2, q) \), each point lies on \( q + 1 \) lines, and each line contains \( q + 1 \) points. The graph \( B(q) \) has \( 2(q^2 + q + 1) \) vertices and degree \( q + 1 \). The graph \( B(q) \) has diameter 3, which will be reduced to diameter 2 by a polarity construction in the next section.

2) Decreasing the Diameter Using a Polarity Map

For each point \([a] \) in \( \text{PG}(2, q) \), the dual \([a]^\perp \) is the line in \( \text{PG}(2, q) \) which contains all points \((x_1, x_2, x_3)\) so that \(a_1x_1 + a_2x_2 + a_3x_3 = 0 \). Since the dual map is a bijection, this shows that \( \text{PG}(2, q) \) contains \( q^2 + q + 1 \) lines. The dual of a line can be defined symmetrically. Notice that \([a]^\perp = [a] \). Clearly, \([x] \) lies on line \([a]^\perp \) if and only if \([a]\) lies on \([x]^\perp \). Such a bijection is also known as polarity.

In order to decrease the diameter of \( B(q) \), we use this polarity: take \( B(q) \) and glue the vertices \( v \in U \) and \( w \in V \) together if and only if \([w] = [v]^\perp \). That is, combine the point \([v] \in U \) and the line \([w] \in V \) together if they are duals of each other. Define \( ER_q \) to be the graph formed by applying this gluing process to \( B(q) \). \( ER_q \) has \( q^2 + q + 1 \) vertices since we glued pairs of vertices in \( B(q) \) together. The degree is still \( q + 1 \), since every line passing through point \([v]\) is glued to a point on \([v]^\perp \), but now the diameter is reduced to 2.

This construction is quite general: if a polarity map exists on a bipartite graph with \( N = 2n \) vertices, maximum degree \( k \), and diameter \( D \), it can be used to construct another graph with \( n \) vertices, maximum degree \( k \), and diameter \( D - 1 \).

3) Quadric Vertices

In a finite geometry, the point \([a]\) may lie on its own dual, the line \([a]^\perp \). When this occurs, i.e., when \(a_1^2 + a_2^2 + a_3^2 = 0 \), the vertex \([a]\) is called a quadric vertex. In \( ER_q \), there is a loop at vertex \([a]\) since \([a]\) and \([a]^\perp \) are glued together. These quadric vertices are the same as the self-orthogonal vectors discussed at the end of Section IV-B, and will be discussed further in Section IV-F in terms of the layout of the network.

F. Structural Properties of ER polarity graphs

We make heavy use of the structure of ER graphs in the design of the network discussed in this paper.

ER graphs \( ER_q \) have \( N = q^2 + q + 1 \) vertices, degree \( k = q + 1 \), and diameter \( D = 2 \). The vertex set of \( ER_q \) can be divided into three disjoint subsets [43]:

- \( W(q) \) set of \( q + 1 \) quadric vertices.
• \( V_1(q) \) → set of \( \frac{q(q+1)}{2} \) vertices adjacent to \( W(q) \).
• \( V_2(q) \) → set of \( \frac{q(q-1)}{2} \) vertices not adjacent to \( W(q) \).

The following properties used in the construction and analysis of the network were presented by Bachratý and Širáň in [41].

**Property 1.** [41] For every odd prime power \( q \), \( ER_q \) has the following properties:

1. No two vertices in \( W(q) \) are directly connected. Every vertex in \( W(q) \) is adjacent to exactly \( q \) vertices in \( V_1(q) \).
2. Every vertex in \( V_1(q) \) is adjacent to exactly 2 vertices in \( W(q) \), and \( \frac{q-1}{2} \) vertices each in \( V_1(q) \) and \( V_2(q) \).
3. Every vertex in \( V_2(q) \) is adjacent to exactly \( \frac{q+1}{2} \) vertices each in \( V_1(q) \) and \( V_2(q) \).
4. There is exactly one path of length two between every vertex pair (considering the self-loop of the self-adjacent quadric as an edge).
5. As a corollary, the edges incident with quadrics vertices do not participate in any triangle. Any edge incident with two non-quadric vertices participates in exactly one triangle.

V. POLARFLY LAYOUT

Network layouts need a modular topology decomposable into smaller units for easy and cost-effective deployment. Since ER graphs are derived from polarity quotient graphs of finite projective planes, such structures are not trivially available, in contrast to topologies derived from multiple generating sets, such as Slim Fly [26] or Bundlefly [47], in which modular units can be readily obtained from individual generators.

Instead, we use the connectivity of quadrics to other vertices to obtain a **modular** and **generalized** layout for PolarFly. Property 1 from Section IV-F tells us that every quadric \( v \in W(q) \) is connected to exactly \( q \) vertices in \( V_1(q) \). We use these \( q \) vertices to construct \( q \) clusters, plus the quadrics themselves as the \((q + 1)^{st}\) cluster.

More formally, Algorithm 1 assigns the vertices in \( ER_q \) into \( q+1 \) clusters, which corresponds to assigning nodes to racks in PolarFly. We use the terms **racks** and **clusters** interchangeably.

For brevity, we only discuss \( ER_q \) for odd \( q \) (even radix) as even prime powers \( q = 2^i \) are sparse in the set of all prime powers. The layout for even \( q \) is similarly modular, and is derived using an analogue to Property 1 for even \( q \) [41].

### Algorithm 1 PolarFly layout

\[ ER_q \leftarrow \text{ER Graph of max degree } q + 1 \]

1. Initialize empty clusters (racks) \( C_0, C_1, \ldots, C_q \)
2. Add all quadrics \( W(q) \) to \( C_0 \)
3. Select an arbitrary quadric \( v \in W(q) \)
4. for each vertex \( u \) adjacent to \( v \)
5. Add \( u \) to an empty cluster \( C_i \)
6. Add all non-quadric neighbors of \( u \) to \( C_i \)

Figure 5 is a diagram of the layout, and Figure 6 shows an example of the \( ER_q \) graph structure supporting this layout.

**Proposition V.1.** Algorithm 1 adds every vertex in \( ER_q \) to exactly one cluster.

**Proof.** Each vertex is at most 2 hops away from the quadric \( v \) selected in Algorithm 1 line 3. The algorithm assigns clusters to \( W(q) \) and all non-quadrics at shortest distance \( \leq 2 \) from \( v \). Thus, every vertex is added to at least one cluster. By Property 1.5, the vertices adjacent to \( v \) are independent and have no common neighbor aside from \( v \). Hence, non-quadric vertices are added to at most one cluster. The quadrics are added to exactly one cluster, \( C_0 \).

A. Intra-rack Layout

1) The quadrics cluster

The layout of the quadrics cluster is quite simple: there are no edges within \( C_0 \) by Property 1.1. This is seen in \( ER_7 \) in Figure 6(a), and also in \( ER_3 \) in Figure 4.

2) Non-quadrics clusters

The layout of a non-quadrics cluster is also rather simple: the edges of a non-quadrics cluster form a fan made up of \( \frac{q-1}{2} \) triangles. The fan has a center vertex, and centers of all fans have a common quadric neighbor. The triangles share only the center, and are otherwise disjoint, giving a fan-blade appearance. This is easily seen in \( ER_7 \) in Figure 6, and may be traced out in \( ER_3 \) in Figure 4.

**Proposition V.2.** The vertices of a cluster form \( \frac{q-1}{2} \) triangles, all having the center of the cluster as a common vertex.

**Proof.** Let \( v \) be the quadric chosen as a starter. Then every vertex \( c_i \) adjacent to \( v \) is selected as the center of cluster \( C_i \). By Property 1.2, \( c_i \) has \( q - 1 \) non-quadric neighbors, and with \( c_i \), these make up the \( q \) elements of the non-quadric cluster \( C_i \). So there is an edge between \( c_i \) and any other vertex in \( C_i \).

If \( u \) is one of the non-center vertices, Property 1.5 tells us that the edge \((c_i, u)\) is contained in exactly one triangle in \( C_i \). This shows that \( C_i \) consists of exactly \( \frac{q-1}{2} \) edge-disjoint triangles, each of which contains \( c_i \).

This fan structure gives PolarFly a modular layout with isomorphic structures induced by all \( q \) non-quadrics clusters. In physical terms, this means that the \( q^2 \) non-quadric nodes can be deployed as \( q \) identical copies of the same rack. Moreover, the fan layout for PolarFly is generalizable to any odd prime power \( q \), unlike Slim Fly where generator sets and thus the intra-rack layout can change significantly with router radix.
Cor or there are multiple $2$.

To prove Proposition V.3.3, it is sufficient to show that no quadric

then $|S(w)| = q - 3$, and if $w \in V_2(q, C_i)$, then $|S(w)| = q - 1$. Every edge in $S(w)$ must connect $w$ to a unique cluster, otherwise there will be multiple 2-hop paths from $w$ to the center of a cluster. This proves Proposition V.4.1.

We give a brief sketch of the proof for Proposition V.4.2 for brevity. Adding $|S(w)|$ for all vertices $w \in C_i$, we see that there are $(q - 1)(q - 2)$ edges between $C_i$ and other non-quadric clusters. If Proposition V.4.2 is false, then there exists a non-quadric vertex at least 3-hops away from the center $c_i$.

This is a contradiction since $ER_q$ has diameter 2.

From Propositions V.4.1 and V.4.2, we see that there are $(q - 3)/2$ independent edges between $V_1(q, C_i)$ and $C_j$. None of these edges are incident to $c_j$. Further, $|V_1(q, C_i) \setminus \{c_i\}| = \frac{q - 1}{2}$. Hence, there must be exactly one $u' \in V_1(q, C_i) \setminus \{c_i\}$ which is not adjacent to $C_j$. Vertex $u'$ can be identified easily as it shares a common quadric neighbor with center $c_j$.

Propositions V.3 and V.4 show that there are $q - 2$ links between all $\frac{q(q - 1)}{2}$ pairs of non-quadric racks, and $q + 1$ links between quadric rack and each of the $q$ non-quadric racks. Thus, PolarFly exhibits a nearly balanced all-to-all connectivity between racks. Moreover, links between any pair of racks can be bundled into cost-effective solutions such as a single multi-core fiber [47].

Using $ER_q$ as an example, Proposition V.3 may be seen in the red edges of Figure 6(b), and Proposition V.4 may be seen in the green and blue edges of Figure 6(b).

C. Triangles and Other Polygons in PolarFly

PolarFly has many triangles, and no quadrangles. The absence of quadrangles is because each pair of non-quadric vertices has exactly one path of length 2, by Property 1.4. If PolarFly had a quadrangle, there would then be vertices with two paths of length 2 between them.

We discuss the abundance of triangles and the implications of this throughout the rest of this section. This becomes important when we analyze the path diversity of PolarFly in the next section.

1) The Number and Types of Triangles in PolarFly

Triangles are of two kinds: those entirely internal to a non-quadric cluster, and those linking three distinct non-quadric clusters together.

Proposition V.5. There are $\binom{q + 1}{3}$ triangles in $ER_q$.

Proof. By Property 1.5, edges not incident to a quadric participate in one triangle, and edges incident to a quadric participate in no triangles. There are $\frac{q(q + 1)}{2}$ total edges, and $q(q + 1)$ edges incident to a quadric. Thus

$$\frac{(q + 1)q(q - 1)}{2}$$

edges are not incident to a quadric. This implies that there are

$$\frac{(q + 1)q(q - 1)}{6} = \binom{q + 1}{3}$$

triangles.
Proposition V.6. The triangles of $ER_q$ either join three distinct non-quadric clusters or are internal to a single non-quadric cluster. In particular,
(a) $\binom{q}{3}$ triangles join non-quadric clusters.
(b) $\binom{q}{2}$ triangles are internal to non-quadric clusters.

Proof. There are a total of $\binom{q}{2}(q - 2)$ inter-cluster edges across non-quadric clusters. From Property 1.4, each such edge participates in exactly one triangle. So there are
$$\frac{q(q - 1)(q - 2)}{2 \cdot 3} = \binom{q}{3}$$
such inter-cluster triangles. Each of these must be made up of three distinct clusters: if not, then the edge $(a,b)$ internal to a cluster will also be on a triangle entirely internal to that cluster, thus there will be two length-2 paths between $a$ and $b$, which cannot occur.

There are $\frac{q-1}{2}$ triangles internal to a non-quadric cluster, by Proposition V.2, and there are $q$ such clusters, so there are
$$\frac{q - 1}{2} \cdot \binom{q}{2} = \binom{q}{3}$$
triangles internal to non-quadric clusters.

Finally,
$$\binom{q}{3} + \frac{q}{2} = \binom{q + 1}{3},$$
so this accounts for all triangles in $ER_q$, by Proposition V.5.

2) Intra-cluster triangles
The $\frac{q-1}{2}$ internal triangles in a non-quadric cluster share the cluster center as a triangle vertex, giving the edges of the cluster the form of a triangle fan-out. They pairwise share no other vertices.

If $q \equiv 1 \mod 4$, the vertices of each internal triangle consist of the center and either two vertices from $V_1$, or two vertices from $V_2$.

If $q \equiv 3 \mod 4$, the vertices of each internal triangle consist of the center, an element of $V_1$ and an element of $V_2$. This may be seen in Figures 6b and 13.

3) Inter-cluster Triangles and a Block Design on Clusters
The remaining triangles are all inter-cluster triangles. In this section, we prove the following Theorem V.7, which says that every non-quadric cluster triplet is joined by one triangle.

This gives rise to a $3 - (q, 3, 1)$ block design, where the $q$ non-quadric clusters are the points, triangles joining cluster triplets are the blocks, and each set of 3 points appears in 1 block. Block designs are well known combinatorial structures that express symmetries in terms of the points and blocks [48], and are therefore of interest in constructing networks.

Theorem V.7. Every triplet of non-quadric clusters under any cluster layout is connected by exactly one triangle.

The theorem is a consequence of the symmetry between all length-2 paths in $ER_q$ that have a quadric as the intermediate node. This symmetry was first shown in [43], and is restated below as Theorem V.8.

To prove Theorem V.7, it suffices to show that all non-quadric triplets are joined by at most one triangle. The theorem then follows from an application of the pigeonhole principle.

To do this, we show a corollary of Theorem V.8 that expresses a symmetry on non-quadric cluster triplets. A technical lemma exhibits an example of such triplets that are joined by at most one triangle. The symmetry in the corollary then implies that this is true of all non-quadric triplets in $ER_q$.

Theorem V.8. [43, Corollary 5] Let $(s_0, w_0, d_0)$ and $(s_1, w_1, d_1)$ be paths of length 2 in $ER_q$, where $s_i$ and $d_i \in V_i$ and $w_i \in W$, the quadrics cluster. Then there exists some automorphism $\theta$ of $ER_q$ such that $\theta(w_0) = w_1$, $\theta(s_0) = s_1$ and $\theta(d_0) = d_1$.

Corollary V.9. If there exists some non-quadric cluster $X$ such that every non-quadric cluster triplet that includes $X$ is joined by at most one triangle, then every non-quadric cluster triplet is joined by at most one triangle.

Proof. Let $w$ be the starter quadric, and let $X$ be a cluster meeting the above condition. Let $x_c$ be the center of $X$.

We assume that there exists some triplet $(D, E, F)$ of distinct non-quadric clusters joined by more than one triangle, and show a contradiction.

Let $d_c, e_c$ and $f_c$ be the centers of $D, E,$ and $F$ respectively. By assumption, the triplet $(D, E, F)$ is joined by two distinct triangles with vertices $(d_0, e_0, f_0)$ and $(d_1, e_1, f_1)$. Let $Y \neq X$ be an arbitrary cluster with center $y_c$. By Theorem V.8, there is an automorphism $\theta$ of $ER_q$ so that $\theta(w) = w$, $\theta(d_c) = x_c$, and $\theta(e_c) = y_c$. Any automorphism of $ER_q$ preserves edges, so $\theta(f_c)$ is connected to $\theta(w) = w$, making $\theta(f_c)$ the center of some cluster $Z$. Again, $\theta$ preserves edges, so two distinct triangles $(\theta(d_0), \theta(e_0), \theta(f_0))$ and $(\theta(d_1), \theta(e_1), \theta(f_1))$ link $X, Y$ and $Z$.

By the condition on $X$, $(X, Y, Z)$ cannot be a triplet, so $Z$ must be one of $X$ or $Y$. But then the triangle $(\theta(d_0), \theta(e_0), \theta(f_0))$ joins exactly two non-quadric clusters. This contradicts Proposition V.6.

Lemma V.10. Let $X$ be a non-quadric cluster whose center $x$ has form $(1, x_1, x_2)$ as a point in $\mathbb{P}^2(F_q)$. Then any triple of distinct non-quadric clusters that includes $X$ is joined by at most one triangle.

Proof. Let $X$ be as stated, and let $Y, Z \neq X$ be distinct clusters such that the triple $(X, Y, Z)$ is joined by the triangle $(a, b, c)$ with $a \in X$, $b \in Y$ and $c \in Z$. Write each point $r$ as $r = (r_0, r_1, r_2)$, a point in $\mathbb{P}^2(F_q)$. Given centers $x, y,$ and $z$ of $X, Y,$ and $Z$, we will write down a system of equations to count how many possible triples $(a, b, c)$ can exist.

By construction, the points $a, b$ and $c$ are connected to the respective centers $x, y$ and $z$, and to each other in $ER_q$. This implies that
$$a \cdot x = b \cdot y = c \cdot z = a \cdot b = b \cdot c = c \cdot a = 0$$

8
in $F_q$. Because $a \cdot x = 0$ and $x = (1, x_1, x_2)$,
\begin{equation}
0 = -(a_1 x_1 + a_2 x_2).
\end{equation}
Because $a \cdot b = b \cdot y = 0$ and $a \cdot c = c \cdot z = 0$, $(a, b, y)$ and $(a, c, z)$ are two-hop paths. The cross-product derivation of the intermediate vertex of a two-hop path given in (2) from Section IV-D then shows that
\begin{equation}
b = a \times y \quad \text{and} \quad c = a \times z.
\end{equation}
Using $b \cdot c = 0$, the above cross-products may be substituted for $b$ and $c$, giving:
\[(a \times y) \cdot (a \times z) = 0\]
Substituting (3) into this equation gives
\begin{equation}
r_{11} a_1^2 + r_{12} a_1 a_2 + r_{22} a_2^2 = 0
\end{equation}
where $r_{11}, r_{12}$ and $r_{22}$ are constants in $y_i, z_i$ for $i \in \{0, 1, 2\}$. Notice that (4) implies we can determine the entries of the points $b$ and $c$ completely once we determine $a$. Thus we have reduced our problem to understanding the number of solutions to (5) in terms of $a_1$ and $a_2$.

Either $a_2$ is 0 or $a_2$ is invertible. If $a_2 = 0$, then (5) implies $a_1 = 0$, and (3) implies $a_0 = 0$. This is impossible since $(0, 0, 0) \notin P^2(F_q)$. So $a_2$ must be invertible.

Without loss of generality, we may then solve for a vertex of the form $a' = (a_0, a_1', 1)$, since $a'$ may then be multiplied by $a_2$ and then left-normalized to give $a \in P^2(F_q)$.

In that case, (5) reduces to a quadratic polynomial in $a_1'$, which has at most two solutions. However, notice that $a = b = c = w$, the starter quadric, satisfies all of the equations. In this case, $a$, $b$, and $c$ do not form a triangle. This implies there is at most one valid solution to (5) and at most one vertex triplet $(a, b, c)$ that form a triangle between the clusters $X$, $Y$ and $Z$.

We are now ready to prove the main theorem of this section.

**Proof of Theorem V.7.** Let $w = (w_0, w_1, w_2)$ be the quadric vertex that generates the cluster layout. At least one of $w_1$ and $w_2$ is non-zero, since $w$ is non-zero and self-orthogonal.

There is thus at least one vector $x$ having form $(1, x_1, x_2)$ that is orthogonal to $w$ (as can be calculated using the dot product on $w$), and this vector $x$ is the center of some non-quadric cluster $X$. By Lemma V.10, any non-quadric cluster triplet that includes $X$ is joined by at most one triangle.

Corollary V.9 then implies that every non-quadric cluster triplet is joined by at most one triangle. By Proposition V.6(a), the number of inter-cluster triangles is $\binom{n}{3}$, which is the same as the number of non-quadric cluster triplets. The theorem immediately follows by the pigeonhole principle.

4) **Distribution of inter-cluster triangles**

We can further classify the triangles in $ER_q$. The distribution of the inter-cluster triangles is as shown in Table II. The table follows from a simple combinatorial argument.

We note that every element of $V_1$ serves as a center in the two layouts induced by its adjacent quadrics. Since the choice of a particular layout does not affect adjacency at all, triangles remain the same in all layouts, in terms of participating vertices. Any triangle with a participating center is entirely internal to the center’s cluster. So if $q \equiv 1 \pmod{4}$, any triangle holding an element of $V_1$ must be of the form $(v_1, v_1, v_1)$ or $(v_1, v_2, v_2)$. Likewise, if $q \equiv 3 \pmod{4}$, any triangle holding an element of $V_1$ must be of the form $(v_1, v_1, v_2)$. We also know that the total number of inter-cluster triangles for any $q$ is $\binom{q}{3}$.

Clusters are either internal to a cluster, or entirely inter-cluster, as in the proof of Proposition V.6. This also implies that if a triangle is internal to a cluster with a center $c$ in a given layout, it will be entirely inter-cluster for any layout in which $c$ is not a center, in other words, layouts induced by any of the $q - 1$ quadrics not adjacent to $c$.

We assume some particular layout starting with an arbitrary starting quadric, and calculate the inter-cluster triangles for that layout. The triangles of course remain the same for any particular layout.

Case 1: $q \equiv 1 \pmod{4}$. First, we calculate the number of inter-cluster triangles of the form $(v_1, v_1, v_1)$. Choose one of the $q$ non-quadric clusters at random. There are $\frac{q-1}{2}$ non-center $V_1$ elements in the cluster. We choose one of these and call it $w$. Looking at a layout in which $w$ is a center, $w$ participates in $\frac{q-1}{2}$ triangles of the form $(v_1, v_1, v_1)$, of which exactly two are internal to the cluster, so the other $\frac{q-5}{2}$ triangles are entirely inter-cluster. Considering all the non-center $V_1$ elements in all the clusters, we get
\[g(q - 1)(q - 5)\]
triangles, but each is counted six times. So there are
\[\frac{g(q - 1)(q - 5)}{24}\]
triangles in total of the form $(v_1, v_1, v_1)$. We also have that $w$ participates in $\frac{q-1}{2}$ triangles of the form $(v_1, v_2, v_2)$. None of these is internal to the cluster, since all internal triangles of that form are $(c, v_2, v_2)$, where $c$ is the center. Considering all the non-center $V_1$ elements in all the clusters, we get
\[\frac{g(q - 1)^2}{4},\]
but each is counted twice. So there are
\[\frac{g(q - 1)^2}{8}\]
triangles in total of the form $(v_1, v_2, v_2)$. Since
\[\frac{g(q - 1)^2}{8} + \frac{g(q - 1)(q - 5)}{24} = \binom{q}{3},\]

| $q \equiv 1 \pmod{4}$ | $g(q-1)(q-5)$ | 0 | $\frac{g(q-1)^2}{24}$ | 0 |
| $q \equiv 3 \pmod{4}$ | 0 | $g(q-1)(q-1)$ | 0 | $\frac{(q+1)g(q-1)}{24}$ |

**TABLE II: Distribution of inter-cluster triangles, of different forms.** The variable $v_1$ indicates a vertex in $V_1$, and the variable $v_2$ indicates a vertex in $V_2$. |
we have accounted for all of the inter-cluster triangles for \( q \equiv 1 \mod 4 \).

Case 2: \( q \equiv 3 \mod 4 \). First, we calculate the number of inter-cluster triangles of the form \((v_1, v_1, v_2)\). Choose one of the \( q \) non-quadric clusters at random. There are \( \frac{q}{2} \) non-center \( V_i \) elements in the cluster. We choose one of these and call it \( w \). We see by considering a layout in which \( w \) is a center that \( w_0 \) participates in \( \frac{q}{2} \) triangles of the form \((v_1, v_1, v_2)\), of which exactly 1 is internal to the cluster, so the other \( \frac{q}{2} \) triangles are entirely inter-cluster. Considering all the non-center \( V_i \) elements in all the clusters, we get \( \frac{q(q-1)(q-3)}{24} \) triangles, but each is counted twice. So there are \( \frac{1}{2} \) triangles in total of the form \((v_1, v_1, v_2)\). We know that when \( q \equiv 3 \mod 4 \), there are no triangles of the form \((v_1, v_1, v_2)\) nor of the form \((v_1, v_1, v_1)\), so the remaining triangles must be of the form \((v_2, v_2, v_2)\), and there are
\[
\left(\frac{q}{3}\right) - \frac{q(q-1)(q-3)}{8} = \frac{(q+1)(q-1)(q-3)}{24}
\]
of these.

As a corollary of Table II, we have Table III, giving the possible types of the intermediate vertex of an alternative 2-hop path between two adjacent vertices. Such a 2-hop path always exists if neither of the vertices are quadric.

| \( q \equiv 1 \mod 4 \) | \( v_1 \) | \( v_1 \) | \( v_2 \) |
| \( q \equiv 3 \mod 4 \) | \( v_1 \) | \( v_2 \) | \( v_1 \) |
| \( q \equiv 3 \mod 4 \) | \( v_2 \) | \( v_1 \) | \( v_2 \) |

TABLE III: Types of the intermediate vertices in a 2-hop path between two adjacent non-quadric vertices.

VI. EXPANDABILITY

Expandability is crucial in budget-driven scenarios, as discussed in Section III. In an underprovisioned expandable network, unused ports on the nodes can be used to incrementally connect additional nodes to the network. In this section, we show that PolarFly affords incremental expansion and present two methods to accomplish this.

Importantly, these methods do not require rewiring of the existing links. They offer a trade-off across different parameters, as summarized in Table IV. These methods are based on cluster replication in PolarFly, which is defined as follows:

**Definition VI.1.** Given a graph \( G(V, E) \), replication of a vertex cluster \( C \subseteq V \) creates a new graph \( G'(V \cup C', E') \). For every vertex \( v \in C \), there exists a replica \( v' \in C' \) such that in the graph \( G' \):

- For every intra-cluster edge \((v, w) \in E\) between two vertices \( \{v, w\} \in C \), the corresponding replicas \( \{v', w'\} \in C' \) are also adjacent, i.e. \((v', w') \in E'\).
- For every inter-cluster edge \((v, w) \in E\) where \( v \in C \) and \( w \notin C \), the replica \( v' \) is adjacent to \( w \), i.e. \((v', w) \in E'\).

Physically, replication is achieved by simply adding an additional rack of nodes, which has similar intra-rack layout and connectivity to rest of the clusters as its original counterpart. Hence, cluster replication methods (sec.VI-A and VI-B) allow modular expansion without rewiring any of the existing links.

A. Replicating the Quadric Cluster

One way to expand PolarFly is to replicate quadric cluster \( C_0 \) as per Def.VI.1, until the desired scale is reached. After replication, to increase the network radix of quadrics, we directly connect every quadric \( v \in C_0 \) and all of its replicas with each other. It can be shown that every replication of \( C_0 \):

1. Increases the number of vertices by \( q+1 \), while preserving the diameter \( D = 2 \).
2. Increases the degree of quadrics \( W(q) \) (and their replicas) and vertices in \( V_1(q) \) by 1 and 2, respectively.
3. Creates \( q+1 \) edges between the replicated cluster \( C_0' \) and all other clusters.

With this method, using \( n \) additional ports per node, the size of PolarFly can be increased by \( \frac{n(q+1)}{2} \), while keeping the diameter \( D = 2 \). However, new links are only added between quadric nodes and \( V_1(q) \). Hence, a large number of quadric replications can result in a non-uniform degree distribution.

<table>
<thead>
<tr>
<th>Method</th>
<th>Scalability</th>
<th>Degree Distribution</th>
<th>Diameter</th>
<th>Average Shortest Path Length</th>
<th>Rewiring</th>
</tr>
</thead>
<tbody>
<tr>
<td>Replicate Quadrics</td>
<td>( \approx q )</td>
<td>Non-uniform</td>
<td>2</td>
<td>( &lt; 2 )</td>
<td>None</td>
</tr>
<tr>
<td>Replicate Non-Quadrics</td>
<td>( \approx q )</td>
<td>Uniform</td>
<td>3</td>
<td>( &lt; 2 )</td>
<td>None</td>
</tr>
</tbody>
</table>

TABLE IV: Characteristics of Expansion Methods. **Scalability** refers to the increase in number of nodes per unit increase in the maximum network radix.

B. Replicating Non-Quadric Clusters

In this method, we expand PolarFly by replicating non-quadric clusters \( C_{i|r>0} \) in a round-robin order, as per Def.VI.1. The replica of \( C_i \) is labeled \( C_{i+q} \), as shown in Figure 7.

Replicating a non-quadric cluster does not add edges to existing center vertices, which can lead to a non-uniform degree distribution. To mitigate this, we note that for every non-quadric cluster \( C_i \) where \( i \neq j \) (and replica \( C_{i+q} \) if it exists), there is exactly one vertex in \( C_i \) with no edges to
We connect the replica of this vertex with the centers of \( C_j \) (\( C_{q+j} \), respectively). It can be shown that for any \( n \leq q \), \( n \) such replications of non-quadric clusters:

1) Increase the number of vertices by \( qn \), which is approximately \( 2^n \) compared to quadric replication.
2) Increase the maximum degree by \( n+1 \).
3) Increase diameter to \( 3^n \) – for every vertex \( u \in C_{q+i} \) (replica \( u' \in C_{i+q} \)), there are at most \( q-1 \) vertices, all in replica \( C_{i+q} (C_i \), respectively), that are at a shortest distance of 3 from \( u \) (\( u' \), respectively).

With non-quadric replication, new links are distributed across all vertices, providing a near-uniform degree distribution. While the diameter of topology increases to 3, the average shortest path length is still clearly less than 2.

VII. ROUTING

To facilitate the adoption of PF, we rely on established schemes and show in the evaluation (Section VIII) that they deliver high performance. However, to show the highest PF potential, we also develop a new adaptive protocol suited specifically for PF. Note that under co-packaged setting, nodes and routers are the same entity in direct networks.

A. Minimal Static Routing

With minimal static routing, a packet is routed from its source router \( R_s \) over the minimal path to its destination router \( R_t \).

B. Valiant Routing

Let \( R_s \) and \( R_t \) denote the source and destination routers, respectively. For each packet, the Valiant routing scheme \([49]\) selects a random router \( R_r \) such that \( R_r \neq R_s \) and \( R_r \neq R_t \). Then, it routes the packet from \( R_s \) to \( R_r \) and \( R_r \) to \( R_t \) along the corresponding shortest paths. This avoids potential hotspots in the network, but reduces the available bandwidth.

The general Valiant design selects some intermediate router. For PolarFly, we use a variant which we call Compact Valiant, where \( R_s \) is chosen from the neighborhood of \( R_s \). The path length for any packet in Compact Valiant is at most 3-hops, as opposed to 4-hops in general Valiant. This reduces the amount of bandwidth wasted on links due to intermediate traffic.

However, the 3-hop route selection will be disadvantageous if the shortest path between \( R_r \) and \( R_t \) goes through the source router \( R_s \). In this scenario, the random neighbor selection would result in the packets bouncing back to the source router. Fortunately, in PolarFly, this situation is easily avoided as it occurs only when \( R_s \) and \( R_t \) are adjacent. Hence, we use Compact Valiant only when \( R_s \) and \( R_t \) are not adjacent.

C. Adaptive Routing

In adaptive routing, the router into which a packet is first injected decides whether this packet should be routed over a minimal path or over a Valiant path. This decision is made on the basis of occupancy of local output buffers used in the respective paths, as well as the lengths of the considered paths. This routing algorithm is called Universal Globally-Adaptive Load-balancing (UGAL) \([50]\).

For PolarFly, we also explore a UGAL variant which we call UGAL\(_{PF} \). To achieve high bandwidth, UGAL\(_{PF} \) reduces the average hops per packet by using:

- Compact Valiant described in Section VII-B, and
- Adaptation threshold – Valiant path is chosen over min-path only when fractional occupancy of the output buffer towards min-path is greater than a threshold (\( \frac{2}{3} \) in our case).

Thus, UGAL\(_{PF} \) offers a trade-off between adaptability of UGAL and low hop count of minimal static routing.

VIII. PERFORMANCE ANALYSIS

We now evaluate the latency and throughput of PolarFly.

A. Methodology and Comparison Targets

We compare PolarFly to Slim Fly \([26]\) (as the most competitive diameter-2 network), Dragonfly \([34]\) (as a popular recent choice when developing interconnects), Jellyfish \([51]\) (as a random expander network) and 3-level fat tree \([37]\) (as the most widespread existing interconnect baseline). Except fat tree, all topologies are direct. As numerous past works illustrate, networks such as torus, hypercube or Flattened Butterfly are less competitive in latency and bandwidth \([26], [34], [52]\).

We use two variants of Dragonfly – (a) balanced Dragonfly (DF1), and (b) Dragonfly with radix and scale almost equivalent to PolarFly (DF2). Configurations of the baseline topologies are given in Table V.

<table>
<thead>
<tr>
<th>Network</th>
<th>Parameters</th>
<th>Number of Routers</th>
<th>Network Radix</th>
</tr>
</thead>
<tbody>
<tr>
<td>PolarFly (PF)</td>
<td>q=31, p=16</td>
<td>993</td>
<td>32</td>
</tr>
<tr>
<td>Slim Fly (SF)</td>
<td>q=23, p=18</td>
<td>1058</td>
<td>35</td>
</tr>
<tr>
<td>Balanced Dragonfly (DF1)</td>
<td>a=12, b=6, p=6</td>
<td>876</td>
<td>17</td>
</tr>
<tr>
<td>Equivalent Dragonfly (DF2)</td>
<td>a=6, b=27, p=10</td>
<td>978</td>
<td>978</td>
</tr>
<tr>
<td>Jellyfish (JF)</td>
<td>–</td>
<td>993</td>
<td>32</td>
</tr>
<tr>
<td>Fat Tree (FT)</td>
<td>n=3, k=18</td>
<td>972</td>
<td>36</td>
</tr>
</tbody>
</table>

Following traffic patterns are simulated to effectively analyze the network behavior:

1) Uniform random traffic – for each packet, the source selects a destination uniformly at random (representing graph processing and distributed-memory graph algorithms, sparse linear algebra solvers, and adaptive mesh refinement methods \([26], [53]-[58]\)).

2) Tornado traffic – endpoints on every router \( i \) send all traffic halfway across to endpoints on router \( i + \frac{N}{2} \) modulo \( N \).

3) Random permutation traffic – a fixed permutation mapping of source to destination is chosen uniformly at random from the set of all permutations. In PolarFly, Tornado and Random permutation traffic are adversarial for min-path routing because there is only one shortest path between any pair of routers.

4) Finally, two special permutation traffic patterns Perm1Hop and Perm2Hop are chosen to analyze UGAL\(_{PF} \). In Perm1Hop, every router communicates with a 1-hop neighbor – the min-path length is 1-hop and valiant path length in UGAL\(_{PF} \) is 4-hops. In Perm2Hop, every router communicates with a 2-hop neighbor – the min-path length is 2-hop and valiant path length in UGAL\(_{PF} \) is 3-hops.
We use the established BookSim simulator [59] to conduct cycle-accurate simulations. Each router along with all of its endpoints in BookSim represents a single co-packaged node. To mimic co-packaged setting under permutation traffic, we enforce that all endpoints of a router send data to endpoints of only one other router. In other words, permutations are computed between routers, and not endpoints. Packets of size 4 flits each are injected with a Bernoulli process. We use input-queued routers with 128 flit buffers per port and 4 virtual channels. In all simulations, we use a warm-up phase where no measurements are taken, to ensure that the simulator first reaches a steady-state.

B. Discussion of Results – Comparison against Baselines
Figure 8 compares the performance of PolarFly (PF) and the topologies shown in Table V. The labels follow the scheme <network>-<routing>. The offered load in Figure 8 is normalized to the maximum capacity of each network.

For Permutation traffic, min-path routing in direct networks can achieve at most 1/p of peak throughput, because all p endpoints of a source router access the same path to the destination router. Hence, we only compare adaptive routing performance under permutation patterns in these topologies.

In general, we observe that PolarFly offers superior performance – for all traffic patterns, it outperforms all competitive direct topologies. Its advantages over Jellyfish and Dragonfly in terms of lower latency, are a direct consequence of its low diameter. Its benefits over Slim Fly in terms of higher saturation bandwidth, are due to careful design of routing protocols that exploit PolarFly structure to ensure that the routing decisions are as good as possible. Amongst the Dragonflies, the balanced DF1 outperforms DF2 whose throughput is bottlenecked by the traffic volume on intra-group links.

For the Uniform traffic, the adaptive routing based on Compact Valiant (UGAL_{PF}) exhibits latency and saturation throughput comparable to that of min-path routing, while significantly outperforming other adaptive algorithms and baseline topologies. Remarkably, the maximum throughput sustained by PolarFly for uniform traffic is comparable to the fat tree, with considerable reduction in latency.

For Random and Tornado Permutation traffic patterns, PolarFly is able to sustain up to 50% of the full injection bandwidth, using adaptive algorithms UGAL and UGAL_{PF}. The performance of these patterns is similar to Perm2Hop traffic shown in figure 9a, as min-path for most packets is 2-hops long. The total buffer space in the min-path is higher compared to Perm1Hop (Figure 9b, all 1-hop min-paths), rendering UGAL_{PF} slower to adapt to congestion. Hence, UGAL_{PF} has considerably higher latency than UGAL for Tornado, Random and Perm2Hop permutation patterns. UGAL has relatively higher entropy in terms of path selection, resulting in smaller queues inside routers and lower latency. Figure 9 also provides detailed insight into adversarial nature of permutation patterns for min-path routing in PolarFly. It can only withstand 5% of the full injection bandwidth, compared to almost 50% with adaptive routing.

C. Discussion of Results - PolarFly Size
Next, we investigate the impact of PolarFly size on the performance by (a) varying q (radix), and (b) expanding network incrementally using the methods described in Section VI. We analyze balanced variants of PolarFly topology under uniform traffic i.e. the ratio of number of endpoints to network radix is maintained to 1 : 2 in all experiments.

Figure 10 shows the latency and throughput for PolarFly for q = 13, 19, 25 and 31, which corresponds to 183, 381, 651 and 993 routers, respectively. The labels follow the scheme <network>_{q}<routing>. All PolarFlies provide similar saturation bandwidth and latency for both min-path and UGAL_{PF} routing. This shows that PolarFly performance is stable with
respect to the size of the network.

Figure 11 shows the latency and throughput of PolarFly incrementally expanded by adding 3, 6, 9 and 12 clusters by quadric or non-quadric cluster replication, which corresponds to approximately 10%, 19%, 29% and 39% increase in network size, respectively. The labels of incrementally expanded networks follow the scheme <network><size increase in percent>-<routing>. We observe that 39% incremental growth in size using quadric replication results in a 31% drop in throughput. Comparatively, non-quadric replication creates only 19% drop in throughput for an equivalent increase in network size, thanks to its near-uniform degree distribution. Moreover, after the first replication, subsequent non-quadric replications have little impact on maximum throughput - 73% of peak bandwidth with 10% incremental growth vs 67% of peak bandwidth with 39% incremental growth.

IX. STRUCTURAL ANALYSIS

We compare bisection bandwidth and link failure resilience of PolarFly, against the topologies given in Table V.

A. Bisection bandwidth

Figure 12 shows the bisection bandwidth of compared topologies in terms of the fraction of edges in the bisection cut-set computed by METIS [60]. Fat Trees provide optimal bisection bandwidth with 50% edges lying in the cut-set. PolarFly closely approximates the optimal ratio, reaching it asymptotically. For network radix $\geq 18$, PolarFly has more than 40% links crossing the bisection, even surpassing random expander networks such as Jellyfish [51]. This is not surprising since PolarFly topology expands extremely well, enforcing an almost Moore Bound spanning tree view from each vertex, whereas Jellyfish relies on random distribution of links and only achieves 50% of links in expectation for a random bisection. PolarFly has significantly higher bisection bandwidth compared to deterministic topologies SlimFly and Dragonfly, that have only 33% and 17% links in bisection.

B. Fault Tolerance and Path Diversity

On the topology configurations given in Table V, we simulate 100 random link failures until network disconnection, and compute the median disconnection ratio.1 We then randomly select a run with median disconnection ratio, and report its variation in network diameter and average shortest path length in Figure 14. We also analyze path diversity in PolarFly in Table VI to better understand its behavior under link failures.

<table>
<thead>
<tr>
<th>Path length</th>
<th>Conditions</th>
<th>Number of paths</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$v, w$ adjacent</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>$v, w$ adjacent and one of $v, w$ quadric, all other cases</td>
<td>0, 1</td>
</tr>
<tr>
<td>3</td>
<td>$v, w$ adjacent, $v, w$ not adjacent, $v, w$ not quadric</td>
<td>$q - 1$</td>
</tr>
<tr>
<td>4</td>
<td>$v, w$ adjacent and neither of $v, w$ quadric, $v, w$ adjacent and one of $v, w$ quadric</td>
<td>$(q - 1)^2$</td>
</tr>
</tbody>
</table>

1Mean and Standard Deviation statistics cannot be used because if any run disconnects at a particular failure ratio, its diameter becomes infinite.
PolarFly, being a random expander, is highly resilient to link failures – random failures in Jellyfish result in just another random graph. PolarFly and SlimFly exhibit similar resilience, with higher disconnection ratio than both Fat tree and balanced Dragonfly DF1. They are both expanders and have comparable resilience to Jellyfish. However, being diameter-2 networks with close to Moore bound scalability, the diameter of PolarFly and SlimFly increases more rapidly than Jellyfish. Compared to PolarFly, SlimFly has slightly more redundancy in minimal paths, resulting in marginally higher disconnection ratio, even though it reduces scalability.

If a single link fails, the diameter of PolarFly increases to 3, or 4 if the link is from a quadric. Table VI shows that there are no 2- or 3-hop paths between quadrics and the adjacent vertices, which intuitively explains why PolarFly diameter increases to 4 with only 5% link failure, as in Figure 14. However, PolarFly has a great deal of path diversity for path length 4, so its diameter stays at 4 even when 55% links fail.

If a node $x$ fails, PolarFly diameter would increase from 2 to 3, as the 2-hop minimal paths between neighbors of $x$ would be lost. However, for any neighbor $v$ of $x$, the neighbors of $v$ have 1-hop or 2-hop paths to other neighbors of $x$, that do not pass through $x$. Hence, despite $x$ failing, $v$ can still reach other nodes within 3-hops.

**X. Cost Analysis**

We now analyze the cost of the network topologies under iso-injection bandwidth constraints. We fill focus on a specific case which is reflective of the latest technological developments: co-packaged Optical IO (OIO) [16]. The primary cost indicator is the total number of optical IO ports: each port requires an OIO module, a laser, a connector and cables. Technological constraints limit to the number of OIO modules that can be co-packaged in a die due to shoreline limitations: the state of the art is 4 to 6 OIO modules per die, with 8 links per module.

We consider configurations with approximately 1,024 nodes, with each topology having the same injection bandwidth. Given that not all the constructions have exactly that number of nodes, we normalize the number of links to a network configuration with 1,024 nodes. In addition, we also consider the achievable injection performance and we normalize the achievable performance, under two distinct scenarios: uniform and permutation traffic. While most networks reach comparable saturation points with uniform traffic, typically around 90%, fat trees are almost insensitive to the type of permutation while direct topologies must resort to some type of misrouting, bringing their saturation points down to approximately 50%. Both PolarFly and Slim Fly use 4 OIO modules with 32 links per node, while Dragonfly 6 OIO modules with 48 links. Fat trees use switches with 4 OIO modules and 32 links, and each of the 1,024 nodes has 2 OIOs with 16 injection links.

Figure 15 shows the relative costs to PolarFly under the two traffic patterns. Slim Fly has a slight cost increase of about 20%, reflective of the lower fraction of Moore’s bound, while the Dragonfly is a diameter 3 network, so the ratio injection bandwidth to overall bandwidth is 1.3 vs 1.2 for PolarFly and Slim Fly. Due to packaging limitations, fat tree switches can only connect two input nodes with 16 links each, resulting in a rather deep 10-level construction of 512 switches per level, and 256 switches in the top level. PolarFly compares very favorably to fat trees with a 5.19X cost reduction under
uniform traffic and 2.68X under permutation traffic.

**Fig. 15:** Cost per node under different topologies normalized to 1,024 nodes.

### XI. Related Work

Network topologies considered in this paper are described in detail in Section II, more details are also provided in a recent survey [61]. Early works into novel topologies with diameter lower than that of 3-stage Fat trees [37] include Flattened Butterfly [52] and its generalization called HyperX [35], and the Dragonfly topology [34], [62]. These designs mainly aimed at facilitating the physical layout of networks. Lowering the diameter of a network in order to reduce cost and power consumption while maintaining high performance have been introduced in the Slim Fly class of interconnects [26], [63]. Since then, several other designs followed, including Xpander [9], Megafly [64], Bundlefly [47] or Galaxyfly [65]. However, they do not focus on diameter-2 and thus none of them improves upon key properties such as latency, cost, or power consumption. PolarFly extends this line of work by exploiting a family of graphs that is asymptotically optimal with respect to the Moore Bound, allowing close to optimal scalability. It simultaneously offers superior cost, power consumption, and performance. Moreover, it specifically targets the recent developments into copackaged optics, something not addressed so far in the literature for scalable network design.

Routing in low-diameter networks has also been a subject of research, especially in recent years. For example, the FatPaths [66] routing architecture, enables adaptive multipathing in data-center and HPC clusters in low-diameter networks, focusing on Slim Fly. However, none of these works is particularly well suited for the unique structure of PolarFly in which some routers form intra-connected clusters while a single cluster of quadric routers forms an independent set. We address this with a novel adaptive UGAL routing protocol suited for PolarFly.

The mathematical foundations of the Erdős-Rényi polarity graphs (ER) are embedded in projective geometry and were laid down in mid-20th century. Singer [67] first formulated perfect difference sets – a numerical structure that encodes the incidence between lines and points of projective planes. Erdős and Rényi [32] discovered the polarity quotient graph of this incidence structure, which forms the basis of PolarFly. Independently, Brown [33] also constructed the same graph using orthogonality relationship of points in projective planes.

Building on these foundations, some prior works have proposed an ER graph topology for high performance interconnection networks. Parhami et al. [68] use perfect difference sets to construct the bipartite network of same degree, diameter and order as the incident graph described in Section IV-E1. Brahme et al. [69] rediscover the ER graphs by defining a symmetric adjacency equation on the perfect difference sets. They also compare the performance of certain communication primitives on this topology and the Clos network. Camarero et al. [70] use the polarity-map based construction of ER graphs (section IV-E2) and compare the cost of conventional networks with various topologies.

To the best of our knowledge, PolarFly is the first work to comprehensively analyze networking properties of ER graphs, covering several aspects beyond the prior attempts [69], [70], including a comparison of feasible radixes, performance for various traffic patterns, bisection width, resilience, and network cost under a co-packaged and iso-bandwidth setting. We also develop a novel modular layout, incremental expansion strategies, and routing schemes to exploit non-minimal path diversity, all of which utilize new mathematical properties of the ER graphs that are presented for the first time in this paper. In this way, our work extends the feasibility of ER graphs as network topologies, well beyond the existing literature.

### XII. Conclusion

In this paper, we propose PolarFly, a diameter-2 network that asymptotically reaches the Moore upper bound on the number of nodes for a given degree and diameter. PolarFly improves upon Slim Fly, being more performant, scalable and cost-effective by up to 10%. Importantly, PolarFly is flexible (it offers a wide range of feasible designs using manufacturable routers), modular (its structure can be decomposed into groups), and expandable (one can incrementally increase its size without much performance loss). We expect that PolarFly will become the enabler for more energy-efficient interconnects in the next-generation era of co-packaged devices.

**Acknowledgment**

The authors would like to thank Guillermo Pineda-Villavicencio for many insightful discussions on the nature of graphs approaching the Moore bound.

**References**


